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# Buckling Optimization of Variable Thickness Composite Plates Subjected to Nonuniform Loads

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**Optimization of heterogeneous composite aerospace structures that operate in a range of temperatures is considered. It is observed that heterogeneous composite plates optimized to operate at a fixed service temperature may perform poorly when that temperature is varied, that is, those plates may be highly sensitive to thermal loadings. A strategy is presented to obtain optimal designs of heterogeneous composite plates that operate within a given temperature range and under arbitrary mechanical loading, such that the significant and undesired sensitivity of these optimal designs to both thermal and mechanical loadings are effectively suppressed. It is assumed that the mechanical loading distribution is arbitrarily piecewise linear and that the operating temperature is within a specified range. Hence, the applied loadings are uncertain or not fixed in nature. Results for heterogeneous composite plates demonstrate the effectiveness of the strategy proposed and the danger of utilizing composite designs sensitive to variations in the thermal and mechanical loadings.**

## Nomenclature

$\boldsymbol{f}_p$ = prebuckling force vector
$\boldsymbol{f}_t$ = thermal load vector
$\boldsymbol{h}$ = vector of reinforcement heights
$h_{2i}, h_{2i-1}$ = outer ply thicknesses
$\bar{h}$ = base plate thickness
$\boldsymbol{K}$ = stiffness matrix
$\boldsymbol{K}_G$ = geometric stiffness matrix
$\boldsymbol{K}_t$ = thermal geometric stiffness matrix
$m$ = number of basis functions
$\boldsymbol{q}$ = eigenvector
$\boldsymbol{q}_p$ = prebuckling displacement vector
$\boldsymbol{q}_t$ = thermal displacement vector
$R_i$ = loading component
$\boldsymbol{r}$ = vector of uncertain loads
$\Delta T$ = temperature difference
$\lambda$ = buckling load

## Introduction

A NUMBER of investigations have been conducted in the past addressing buckling optimization of composite structures. A common feature of these optimization studies is that, in most of them, the applied in-plane loading distribution is uniform and the service temperature of the structural component is constant. Thus, the buckling load magnitude is the objective function to be maximized.

Specification of both mechanical loading distribution and operation temperature may result in optimal designs that present significant sensitivity to variations in the loading configuration. The optimization problem of an isotropic plate with piecewise constant thickness subjected to an arbitrary nonuniform loading distribution was considered in Ref. 1. The loading distribution was modeled as a piecewise linear function, and it was assumed that the prebuckling behavior is temperature independent. However, heterogeneous composite plates are not free of thermal residual stresses inherited from the cure processing at elevated temperatures.[2] The effect of these thermal residual stresses on buckling loads may be quite significant and depends on the difference between service and cure temperature. Note that a composite plate optimized to operate at a specified fixed temperature may perform poorly at another temperature. Therefore, given that aerospace structures actually operate within reasonably wide temperature ranges, the design may be vulnerable to temperature variations within the operating range.

The fundamental idea of this work is to propose an optimization procedure that accounts for variations of both mechanical and thermal loadings within given bounds. It is assumed that the mechanical loading distribution is arbitrarily piecewise linear[1] and that the operating temperature is within a specified range. Therefore, the mechanical and thermal loading distributions are uncertain or not fixed. A detailed explanation of the technique to treat uncertain loads is not the focus of this paper, but its rigorous proof[3] and application to buckling optimization problems is well documented.[1,4]

## Problem Formulation

The design variables of the optimization problem considered are the thicknesses of the composite plate that vary from region to region, the mechanical loading distributions that vary continuously, and the thermal residual stress distribution, whereas the objective function is the critical buckling load. Figure 1a shows an arbitrary piecewise linear loading distribution in a three-dimensional view of a composite plate with variable thickness, where the thicknesses are exaggerated to facilitate visualization.

The strips in Fig. 1a represent regions with different thicknesses. Those regions are referred to as reinforcements due to their discrete nature. A base plate is defined such that the plate thickness at any point cannot be smaller than the thickness of the base plate. The base plate is chosen to be a $[0/90]_S$ laminate; its total thickness is kept constant, and its four layers have equal thicknesses. On the other hand, the outermost layers have variable thicknesses that must be assigned according to laminate symmetry. The stacking sequence can be better visualized in Fig. 1b, where an arbitrary cross section of the plate is shown with ply orientations. It can be inferred that the stacking sequence of reinforcement $i$ is represented by $[0^{h_{2i}}/90^{h_{2i-1}}/0^{\bar{h}/4}/90^{\bar{h}/4}]_S$, where $\bar{h}$ is the base plate thickness, $h_{2i-1}$ is thickness of the outer 90-deg ply, and $h_{2i}$ is thickness of the outer 0-deg ply.

The optimization problem must be formulated to take into account the variability of the piecewise linear loading distribution shown in Fig. 1a, as well as the variable service temperature. This can be done via a minimax formulation as given in Eq. (1) (Ref. 1):

$$\max_{\boldsymbol{h}} \min_{\boldsymbol{r},\Delta T} \lambda(\boldsymbol{h}, \boldsymbol{r}, \Delta T) \tag{1}$$

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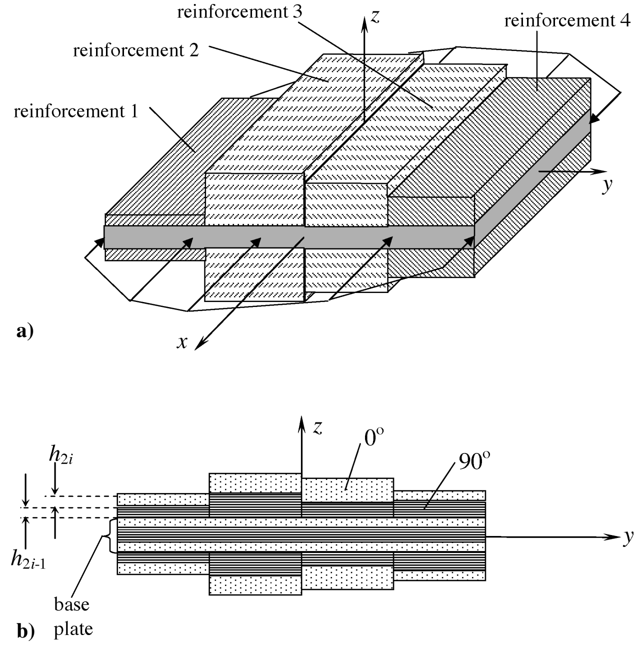


**Fig. 1 Symmetric composite plate.**

where $\boldsymbol{h}$ is the vector of reinforcement thicknesses, $\boldsymbol{r} = (R_1, \ldots, R_m)$ is the vector of uncertain loads when $m$ loading components are considered, $\Delta T$ is the difference between the service and cure temperatures, and $\lambda$ is the buckling load magnitude, that is, the objective function of the problem. The definition of $\boldsymbol{r}$ and the physical significance of its components $R_i$ can be found in Ref. 1. Note, however, that the components $R_i$ hold the following relationship: $\Sigma R_i = 1$ and $R_i \in [0, 1]$. The arbitrary piecewise loading distribution is represented through a series of piecewise linear basis functions defined at specific points along the plate edges. Component $R_i$ is associated with the $i$th basis function and provides a measure of its relative contribution to the overall piecewise linear loading, whereas the basis functions themselves define the shape of the loading distribution. The composite plates optimized have constant volume such that a constraint to the optimization problem exists: summation of the outermost ply thicknesses shown in Fig. 1b must be constant.

Solution of the problem stated in Eq. (1) provides, simultaneously, the optimal design and the worst loading combination in terms of $\boldsymbol{r}$ and $\Delta T$. The considered optimization problem is bilevel, and its numerical solution is often laborious. However, the fact that $\lambda$ is obtained through solution of the classic eigenvalue problem allows for a tremendous simplification if the extended stability boundary theorem[3] is used.

Calculation of the objective function $\lambda$ is divided into three steps: 1) solution of a linear thermal problem to obtain thermal residual stresses, 2) solution of a linear prebuckling problem to obtain prebuckling mechanical stresses, and 3) solution of an eigenvalue problem to obtain the buckling load. Furthermore, the laminate is symmetric and assumed free of initial imperfections, such that a bifurcation type of buckling is observed. The resulting thermal, prebuckling, and buckling problems can be stated, respectively, in a compact form as

$$\boldsymbol{K}\boldsymbol{q}_t = \boldsymbol{f}_t, \qquad \boldsymbol{K}\boldsymbol{q}_p = \boldsymbol{f}_p, \qquad (\boldsymbol{K} + \boldsymbol{K}_t - \lambda \boldsymbol{K}_G)\boldsymbol{q} = \boldsymbol{0} \tag{2}$$

The thermal load vector $\boldsymbol{f}_t$ is proportional to the difference $\Delta T$ between operation and processing temperatures. Accordingly, the thermal geometric stiffness matrix $\boldsymbol{K}_t$ is also proportional to $\Delta T$. For example, given a cure temperature of 120°C and operation temperature of 25°C, both $\boldsymbol{f}_t$ and $\boldsymbol{K}_t$ are proportional to $\Delta T = -95$°C.

The stability boundary theorem is valid provided that $\boldsymbol{K} + \boldsymbol{K}_t$ is positive definite and even when $\boldsymbol{K}_G$ is indefinite.[3] Matrix $\boldsymbol{K} + \boldsymbol{K}_t$ is guaranteed to be positive definite if thermal buckling has not occurred. Because buckling maximization is sought, the optimizer will avoid designs that result in matrices $\boldsymbol{K}_t$ that render $\boldsymbol{K} + \boldsymbol{K}_t$ indefinite, which ensures that the hypothesis of the stability boundary theorem is observed. However, as a safeguard, a Sturm check of $\boldsymbol{K} + \boldsymbol{K}_t$ is conducted for every design evaluated to make sure that positive definiteness is satisfied. This check adds little computational cost to the analysis because matrix $\boldsymbol{K} + \boldsymbol{K}_t$ must be decomposed anyway to obtain the eigenvalues of the problem in Eq. (2c).

Minimization of $\lambda(\boldsymbol{h}, \boldsymbol{r}, \Delta T)$ with respect to $\boldsymbol{r}$ and $\Delta T$ is easily accomplished with the aid of the stability boundary theorem because it suffices to evaluate $\lambda$ at extremal points. Given that the service temperature can vary within a range $\Delta T_{\min} \leq \Delta T \leq \Delta T_{\max}$, these extremal points correspond to $R_i = 1$ and $\Delta T = \Delta T_{\min}$, and $R_i = 1$ and $\Delta T = \Delta T_{\max}$ for $i = 1, \ldots, m$, that is, there are $2m$ extremal points.

The solution of the optimization problem is initiated by assessment of 400 random designs in an attempt to reduce the risk of convergence to local optima. The best design obtained by the random procedure is taken as the starting point for a Powell's search.[5] The eigenvalues of the problem may be nonsmooth, but they are certainly continuous. Hence, an optimization algorithm that requires only $C^0$ continuity of the objective function and constraints was selected (Powell's method). Alternative strategies can be employed to handle the nonsmoothness of the eigenvalues, such that more efficient gradient-based optimization methods can be used to ensure faster convergence to the optimal design.[6] The optimization stops when the relative difference between the previous and present values of $\phi$ does not exceed 0.001, where $\phi$ is the minimum $\lambda$ over the admissible loading configurations $\boldsymbol{r}$ and temperature difference $\Delta T$. In other words, $\phi$ is the solution of the inner loop of the minimax problem stated in Eq. (1).

## Numerical Simulations and Discussion

A square plate simply supported along four edges with sides of 360 mm is investigated, where the loading is applied along the sides parallel to the $y$ axis as shown in Fig. 1a. The material chosen for simulation is graphite/epoxy T300-5208, whose properties are given in Table 1. Four reinforcements are considered, and five piecewise linear basis functions are defined at positions $y = 0$, 90, 180, 270, and 360 mm along the two opposite edges parallel to the $y$ axis. The base plate total thickness is 0.3 mm, such that each base ply is 0.075 mm thick. Moreover, a constant volume constraint is imposed such that $\Sigma(h_{2i-1} + h_{2i}) = 0.6$ mm, $i = 1, 2, 3, 4$. The processing temperature adopted is 120°C, and the service temperature ranges between −60°C and +40°C, such that $-180°\text{C} \leq \Delta T \leq -80°\text{C}$.

Initially, it would be interesting to establish a comparison between the optimal designs obtained under traditional assumptions and the present strategy. For that purpose, it is worth finding a few classical optimal designs for fixed loading distributions and fixed operation temperature of 20°C, for instance, such that $\Delta T = -100$°C. Figures 2a–2c show the optimal plate configurations obtained when individual distributed loading components (basis functions) are applied, and Fig. 2d presents the optimal plate for the uniform loading. The force magnitudes are the optimal net forces caused by the distributed loadings whose shapes are also shown. Because of the symmetry of the basis functions about $y = 180$ mm, only three cases are shown.

Notice that the optimal design obtained for $R_3 = 1.0$ (Fig. 2c) is slightly nonsymmetric with respect to the plate center line $y = 180$ mm, despite the fact that the loading and boundary

**Table 1 Material properties of the T300-5208 graphite/epoxy**

| Property | Value |
|---|---|
| Principal modulus of elasticity $E_{11}$ | 154.0 GPa |
| Principal modulus of elasticity $E_{22}$ | 11.13 GPa |
| In-plane Poisson's ratio $\nu_{12}$ | 0.304 |
| In-plane shear modulus $G_{12}$ | 6.98 GPa |
| Transverse shear modulus $G_{13}$ | 6.98 GPa |
| Transverse shear modulus $G_{23}$ | 3.36 GPa |
| Principal thermal expansion coefficient $\alpha_1$ | $-0.17 \times 10^{-6}$°C$^{-1}$ |
| Principal thermal expansion coefficient $\alpha_2$ | $23.1 \times 10^{6}$°C$^{-1}$ |

**Table 2 Mechanical sensitivity of traditional optimal designs**

| Optimal design | $R_1 = 1$, N | $R_2 = 1$, N | $R_3 = 1$, N | $R_4 = 1$, N | $R_5 = 1$, N | Uniform, N |
|---|---|---|---|---|---|---|
| 1 (Fig 2a) | 895.2 | 235.8 | 2.7 | 1.5 | 2.8 | 3.2 |
| 2 (Fig 2b) | 729.4 | 284.4 | 39.8 | 11.6 | 11.6 | 26.2 |
| 3 (Fig 2c) | 69.3 | 66.0 | 171.9 | 93.6 | 98.4 | 149.0 |
| 4 (Fig 2d) | 174.7 | 142.3 | 139.6 | 142.3 | 174.7 | 270.2 |

**Table 3 Minimax optimal design: buckling loads**

| $\Delta T$ | $R_1 = 1$, N | $R_2 = 1$, N | $R_3 = 1$, N | $R_4 = 1$, N | $R_5 = 1$, N | Uniform, N |
|---|---|---|---|---|---|---|
| −180°C | 139.8 | 127.2 | 161.3 | 127.2 | 139.8 | 262.4 |
| −80°C | 210.9 | 146.5 | 126.0 | 146.5 | 210.9 | 233.3 |

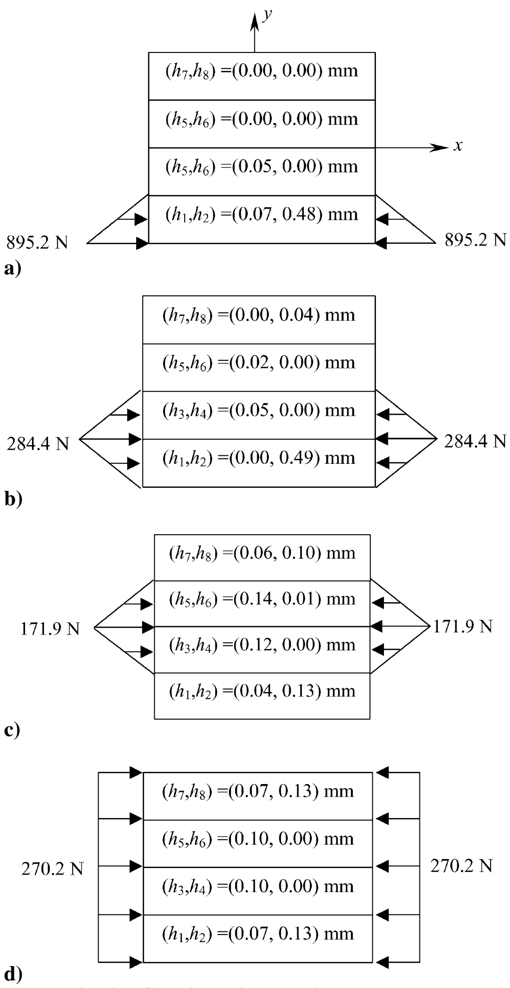


**Fig. 2 Classic optimal designs and loads.**

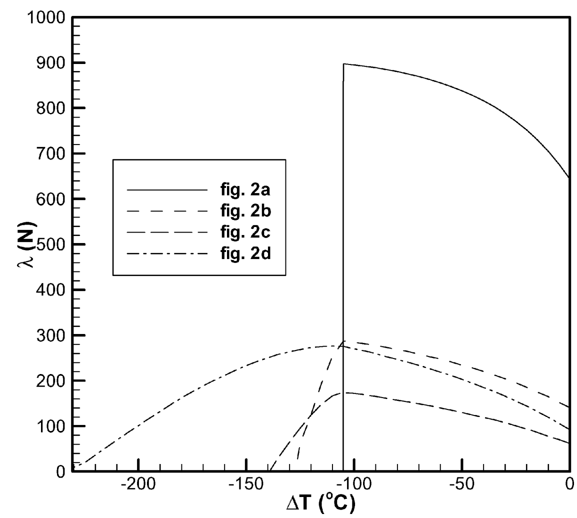


**Fig. 3 Thermal sensitivity of traditional optimal designs.**

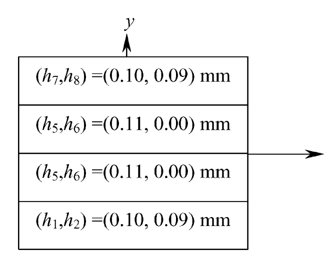


**Fig. 4 Optimal design subjected to uncertain loads.**

conditions are symmetric. This is in accordance with the optimal design that would have been obtained if the material considered were isotropic because, in this case, a pronounced nonsymmetric design is reached.[1] On the other hand, when the uniform loading is applied (Fig. 2d), the optimal design found is perfectly symmetric. This observation justifies the use of full meshes to model the entire plate without any assumption of any kind of symmetry for the optimal designs.

Table 2 presents results concerning sensitivity of the optimal designs obtained for variations in the mechanical loadings. The lines of Table 2 correspond to the optimal designs presented in Figs. 2a–2d and the columns correspond to the loading distribution components applied at a constant temperature difference $\Delta T = -100^\circ$C. Note that all optimal designs perform poorly if the loading configuration applied is other than that particular one for which it was optimized. For instance, the optimal design of Fig. 2a may have its critical load dramatically reduced if an unfavorable loading distribution is applied. The conclusion drawn is that the thermal residual stresses in the optimal design of Fig. 2a lead to potentially unstable plates because very small forces applied at unfavorable locations imply loss of stability. This is obviously a very dangerous design and would not be a good choice in practical situations.

Figure 3 presents the sensitivity of the obtained optimal designs to variations in the thermal loads. The mechanical loading distribution is maintained constant as in the optimal designs presented in Figs. 2a–2d, and the temperature difference $\Delta T$ is varied from 0°C until the point where thermal buckling is observed. High sensitivities to temperature variations are noticed for the optimal designs. The sensitivity is particularly high near $\Delta T = -100^\circ$C because the plates were optimized to operate at this temperature by the use of the traditional approach. As expected, the maximum buckling loads for all plates correspond to $\Delta T$ approximately equal to $-100^\circ$C. In the case of the optimal design related to Fig. 2a, thermal buckling happens when $\Delta T = -105.1^\circ$C, whereas for Fig. 2b, it occurs when $\Delta T = -127.2^\circ$C, both within the plate operating temperature range. The results shown in Fig. 3 clearly demonstrate the danger of optimal designs obtained under fixed thermal loads assumptions.

Figure 4 shows the optimal plate obtained by the minimax strategy. In this case, the critical buckling loads are shown in Table 3 for several different loading configurations. The buckling loads are now much less sensitive to loading distribution than those presented in Table 2 and to the corresponding configurations illustrated in Figs. 2a–2c. For the minimax optimal design presented in Fig. 4, the critical load $\lambda$ is greater than 126.0 N for any loading combination $\boldsymbol{r}$ and $\Delta T$, provided that $\Sigma R_i = 1$, $R_i \geq 0$, and $-180 \leq \Delta T \leq -80$°C. Direct comparison of the minimax optimal design (Fig. 4) and the optimal design shown in Fig. 2d may be misleading because, apparently, the later has better performance over the former by simple observation of Tables 2 and 3. This is not true because buckling at different temperatures must also be assessed to make a fair comparison. In this sense, when $R_2 = 1$ and $\Delta T = -180$°C, the optimal design of Fig. 2d leads to $\lambda = 72.2$ N, well below 126.0 N.

A finite element (FE) mesh of $8 \times 8$ elements was used throughout. Refinement of the plate FE mesh does not significantly alter results because the element used is a bicubic Lagrangian[2] whose complexity (element with 80 degrees of freedom) allows for the use of relatively coarse FE meshes. Finer meshes ($12 \times 12$) have been tried with small variation in terms of buckling loads (less than 5%) and, although they certainly improve accuracy, they also require more computer processing. This is costly because an optimization is performed, and, therefore, computation of eigenvalues is performed thousands of times.

## Conclusions

Buckling optimization of composite plates subjected to piecewise linear loadings is resolved within the scope of a minimax strategy. The thermal residual stresses that are inherited from the thermal processing are included in the analysis. Actually, the optimization procedure obtains optimal designs that take advantage of the thermal residual stresses. If disregarded, those residual stresses will probably impair the performance of composite plates.[2]

Traditionally, aerospace structural components optimized for buckling are ideally subjected to perfectly uniform compressive loads and thermal effects are ignored. However, this mechanical loading configuration is rarely observed in actual service conditions, and thermal effects may strongly affect the performance of composite plates. Therefore, this procedure may result in designs that are not optimal and that may be highly sensitive to variations in the applied loading and temperature. This paper proposes a reformulation of the problem to accommodate the uncertain nature of the mechanical and thermal loadings.

The simple examples investigated demonstrate the danger of high sensitivity to mechanical and thermal loading perturbations by the optimization of composite plates with piecewise constant thickness. It was determined that the worst loading acting on the minimax optimal composite plate in Fig. 4 is the one where $R_3 = 1$ and $\Delta T = -80$°C (or $R_2 = 1$ and $\Delta T = -180$°C) with the associated buckling load $\lambda = 126.0$ N. The optimization problem is posed to guarantee that, if the mechanical loading configuration and/or the service temperature are changed within the admissible set, the buckling load will not decrease, that is, $\lambda > 126.0$ N.

The uncertainty of the loading is assumed to be associated with one direction (the $x$ axis). Consideration of uncertain loadings acting in the $y$ axis of the plate is straightforward because all that one must do is include the applicable proportionality parameters $R_i$ in vector $\boldsymbol{r}$ and enforce the convex combination constraint $\Sigma R_i = 1$. In this situation, it may be more effective to assume that the plate thickness varies in both the $x$ and $y$ directions.

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## References

[1] de Faria, A. R., and de Almeida, S. F. M., "Buckling Optimization of Plates with Variable Thickness Subjected to Nonuniform Uncertain Loads," *International Journal of Solids and Structures*, Vol. 40, No. 15, 2003, pp. 3955–3966.

[2] de Almeida, S. F. M., and Hansen, J. S., "Enhanced Elastic Buckling Loads of Composite Plate with Tailored Thermal Residual Stresses," *Journal of Applied Mechanics*, Vol. 64, No. 4, 1997, pp. 772–780.

[3] de Faria, A. R., and Hansen, J. S., "On Buckling Optimization Under Uncertain Loading Combinations," *Structural and Multidisciplinary Optimization Journal*, Vol. 21, No. 4, 2001, pp. 272–282.

[4] de Faria, A. R., and Hansen, J. S., "Buckling Optimization of Composite Axisymmetric Cylindrical Shells Under Uncertain Loading Combinations," *Journal of Applied Mechanics*, Vol. 68, No. 4, 2001, pp. 632–639.

[5] Powell, M. J. D., "An Efficient Method for Finding the Minimum of a Function of Several Variables Without Calculating Derivatives," *Computers Journal*, Vol. 7, 1964, pp. 155–162.

[6] Andrade, L. H., de Almeida, S. F. M., and Hernandes, J. A., "Optimization of Buckling Loads of Thin Reinforced Orthotropic Plates Including the Effect of Thermal Residual Stresses," *Proceedings of the Third World Congress of Structural and Multidisciplinary Optimization*, Vol. 1, Buffalo, NY, 1999, pp. 138–140.

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